Lattice Bosons

$J.Chakrabarti^{a,1}, A.Basu^a and B.Bagchi^{b,2}$

a. Department of Theoretical Physics

Indian Association for the Cultivation of Science

Calcutta-700032 INDIA

b. Department of Applied Mathematics

University of Calcutta

92 Acharya Prafulla Chandra Road

Calcutta-700009 INDIA

Abstract

Fermions on the lattice have bosonic excitations generated from the underlying periodic background. These , the lattice bosons , arise near the empty band or when

the bands are nearly full. They do not depend on the nature of the interactions and exist for any fermion-fermion coupling .We discuss these lattice boson solutions for the Dirac Hamiltonian.

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1.Electronic address: tpjc@mahendra.iacs.res.in

2.Electronic address: bbagchi@cucc.ernet.in

1 Introduction

The fermions often have boson solutions. These bosons, made usually from the fermion degrees ,exist in many systems [1]. We present in this work the boson solutions that arise from the background lattice.They do not arise from the interactions, but are related to the spatial geometrical features, such as the periodicities, of the underlying lattice on which the fermions reside.Because of this geometrical origin these bosons, we believe, are robust and exist for any value of coupling.These,the lattice bosons,are the subject of this work.

It is worth recalling the many-body fermion states in low dimensions are eigenstates of boson operators.These ideas of bosonisation have received wide support over the years[2-6].

Here, in this work , we illustrate our ideas on the one-dimensional equispaced lattice. We find the fermion algebra that is represented by this lattice.This algebra of fermions shows the curious feature that for low values of fermion filling on the lattice some of the generators behave as bosons. Interestingly, the same is true in the "insulating" region where the fermion states are nearly full.The boson interpretations in these two limits,the nearly empty lattice(nel), and the nearly filled lattice(nfl),are different.The roles of the creation and the destruction operators are reversed.These interpretations do not depend on the underlying Hamiltonian or the nature of the fermion interactions.

The lattice bosons in the nel are made of coherently superposed fermion pairs.In the "insulating " region, i.e. in the nfl,coherently superposed hole pairs create the lattice bosons.The usual Hamiltonian of the fermions is recast in terms of these bosons in the nel and the nfl regions.The new Hamiltonian in the lattice boson variables is diagonalised to get the boson spectrum.

To begin we introduce on the one-dimensional equispaced lattice , with periodic boundary conditions,the set of generators,made of fermionic objects, that close under commutations.This algebraic structure, represented by the background lattice, arises solely from the anticommuting properties of the fermions. It does not depend on any Hamiltonian or on the interactions.We illustrate the algebra on the simple 6-point lattice. Following from this algebraic structure we identify the generators that are bosonic.We show that for low values of fermion filling a set of generators satisfy bosonic commutation rules.As the filling increases the boson approximation gets worse.Interestingly ,near maximum filling , i.e. near the "insulating" domain the boson modes return,but now with the creation and the destruction operators reversing their roles. This leads us to interpret the bosons as coherent superposition of the hole pairs.

With this structure in place we take up a simple Hamiltonian and look for its lattice boson solutions. Our choice is the Dirac Hamiltonian for fermions of mass.

We set up the lattice boson algebra and solve for the boson eigenstates. These are good solutions in the nel and the nfl regions.

2 The Lattice Boson Algebra

Consider the lattice of 2N equispaced points with periodic boundary conditions. Let $C_n^\dagger$ and C_n denote the creation and the annihilation operators of the fermion on site denoted by the index n.Clearly,

$$\{C_n, C_m^\dagger\} = \delta_{nm} \tag{1}$$

$$\{C_n^\dagger, C_m^\dagger\} = \{C_n, C_m\} = 0 \tag{2}$$

Consider the generators,

$$e_{+l} = \sum C_n^\dagger C_{n+l}^\dagger, \tag{3}$$

where l takes values 1,2,3... For the lattice of 6 points the generators read :

$$e_{+1} = c_1^\dagger c_2^\dagger + c_2^\dagger c_3^\dagger + c_3^\dagger c_4 \dagger + c_4^\dagger c_5^\dagger + c_5^\dagger c_6^\dagger + c_6^\dagger c_1^\dagger \tag{4}$$

$$e_{+2} = c_1^\dagger c_3^\dagger + c_2^\dagger c_4^\dagger + c_3^\dagger c_5^\dagger + c_4^\dagger c_6^\dagger + c_5^\dagger c_1^\dagger + c_6^\dagger c_2^\dagger \tag{5}$$

$$e_{+3} = c_1^\dagger c_4^\dagger + c_2^\dagger c_5^\dagger + c_3^\dagger c_6^\dagger + c_4^\dagger c_1^\dagger + c_5^\dagger c_2^\dagger + c_6^\dagger c_3^\dagger \tag{6}$$

and so on.

But ,interestingly, $e_{+3} = 0$, since the last three terms of e_{+3} ,using (2), are negatives of the first three . Further e_{+4} and e_{+5} can also be shown to be negatives of e_{+2} and e_{+1} respectively . So for the lattice of 6(i.e. (2N)) points the number of independent generators(e_{+l}) are 2 (i.e. (N-1)), corresponding to l=1 and l=2. Consider now the conjugates of e_{+l} . Denoted by e_{-l} ,for this case of the 6-point-lattice, these are:

$$e_{-1} = -[c_1c_2 + c_2c_3 + c_3c_4 + c_4c_5 + c_5c_6 + c_6c_1] \tag{7}$$

$$e_{-2} = -[c_1c_3 + c_2c_4 + c_3c_5 + c_4c_6 + c_5c_1 + c_6c_2] \tag{8}$$

Now if we calculate the commutator of e_{+1} and e_{-1} ,we get:

$$[e_{-1}, e_{+1}] = 6 - 2\sum c_n^\dagger c_n + \sum (c_n^\dagger c_{n+2} + h.c.) \tag{9}$$

where we have used the anticommutators (1).The quantities on the r.h.s. of (9)

$$h_0 = \sum c_n^\dagger c_n \tag{10}$$

$$h_2 = \sum (c_n^\dagger c_{n+2} + h.c.) \tag{11}$$

are the fermion number operators, h_0 ,and the hopping operator, h_2.

The quantity 6(i.e.,2N) is just the total number of points on the lattice. Similarly,

$$[e_{-2}, e_{+1}] = -h_1 + h_3 \tag{12}$$

where the h_i is defined as $\sum(c_n^{\dagger} c_{n+i} + h.c.)$; the h_1 and the h_3 are two hopping operators.It is then easy to check that :

$$[e_{\pm l}, h_0] = \mp 2 e_{\pm l} \tag{13}$$

$$[e_{\pm 1}, h_1] = \mp e_{\pm 2}; [e_{\pm 1}, h_2] = \pm e_{\pm 1}; [e_{\pm 1}, h_3] = \pm e_{\pm 2} \tag{14}$$

$$[e_{\pm 2}, h_1] = \mp e_{\pm 1}; [e_{\pm 2}, h_2] = \pm e_{\pm 2}; [e_{\pm 2}, h_3] = \pm e_{\pm 1} \tag{15}$$

Further,

$$[e_{+i}, e_{+j}] = [e_{-i}, e_{-j}] = [h_i, h_j] = 0 \tag{16}$$

for all i and j. Thus $e_{\pm 1}, e_{\pm 2}, h_0, h_1, h_2, h_3$,form a closed algebra for the 6-point lattice. The generalisation to the case of the 2N lattice is straightforward.For this general case the number of independent e_{+l} generators are (N-1). The number of independent h_i generators are (N+1).

The structure of the algebra is:

$$[e_{+l}, e_{-l}] = 2N - 2h_0 + h_{2l} \tag{17}$$

$$[e_{+l}, e_{-l'}] = \sum \alpha^j_{ll'} h_j \qquad (l \neq l') \tag{18}$$

$$[e_{\pm l}, h_0] = \mp 2 e_{\pm l} \tag{19}$$

$$[e_{\pm l}, h_i] = \sum \beta^j_{li} e_{\pm j} \tag{20}$$

$$[e_{+l}, e_{+l'}] = [e_{-l}, e_{-l'}] = [h_i, h_j] = 0 \tag{21}$$

Note that the sums over j in (18) and (20) run over only a small number of j values. That is $\alpha^j_{ll'}$ and β^j_{li} are non-zero only for one or two values of j for given ll' and li.The fermion number h_0 does not appear on the rhs of (18).

3 The Boson Interpretation

Consider now the commutator (17) of

$$[e_{-l}, e_{+l}] = 2N - 2h_0 + h_{2l} \tag{22}$$

where h_0 is the fermion number operator. The first term, 2N , is the size of the lattice. For small

values of fermion filling i.e. when the band is nearly empty h_0 is way small compared to 2N.

Further the operator h_{2l} given by :

$$h_{2l} = \sum (C_n^\dagger C_{n+2l} + h.c.) \tag{23}$$

in its diagonal form reads:

$$h_{2l} = \sum 2cos2klaC_k^\dagger C_k \tag{24}$$

Thus the value of h_{2l} for the state of the single fermion is bounded by ± 2.For the small number of fermions in the lattice(nel) the value of h_{2l} is again way small compared to the first term 2N. Therefore equation (22) in the nel region becomes

$$[e_{-l}, e_{+l}] = 2N \tag{25}$$

For the normalised generators $\frac{1}{\sqrt{2N}} e_{\pm l}$ the r.h.s. of (25) becomes 1.Consider now the commutators $[e_{-l}, e_{+l'}]$ with different l and l' .From (18) we get $[e_{-l}, e_{+l'}] = \sum \alpha_{ll'}^j h_j$ (in the sum $j \neq 0$). The generators h_i are all simultaneously diagonalisable.All of them have eigenvalues bounded by 2 for the single fermion .(See also the discussions following the eqn (21)).For the normalized generators we ,therefore, get:

$$[e_{-l}, e_{+l'}] = 0 \ \ when \ \ l \neq l' \tag{26}$$

Taken together (25) and (26) gives for all values of l and l' the relation:

$$[e_{-l}, e_{+l'}] = \delta_{l,l'} \tag{27}$$

These are therefore the creation and the destruction operators of bosons. Consider now the insulating limit , when the lattice is nearly full(nfl). In the eqn (17) the value of h_0 is about 2N, so that combining the first two terms on the r.h.s. of (17) we get

$$[e_{-l}, e_{+l}] = -2N \tag{28}$$

as for the nfl the value of the h_i are finite ,near zero. Thus if the creation and the annihilation operators are interchanged ,the same bosonic commutator relation reappears. In the insulating limit ,therefore, the coherent superposed pairs of holes created by e_{-l} on the insulator is the lattice boson creation operator.

4 The Dirac Hamiltonian

Consider the Dirac Hamiltonian with mass on the one-dimensional equispaced lattice of 2N sites. The continuum Hamiltonian is[7,8] :

$$H = -i\psi^{\dagger}(\alpha.\partial)\psi \tag{29}$$

with ψ being the two-component wave-function $\binom{\psi_1}{\psi_2}$ and the Dirac matrices of interest are $\gamma_0 = \sigma_3$; $\alpha = \gamma_5 = \sigma_1$ given by:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} and \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{30}$$

respectively. If we choose

$$\psi_{\pm} = \frac{1}{2}(1 \pm \gamma_5)\psi$$

and then write $c = \psi_1 + \psi_2$, $b = \psi_1 - \psi_2$ and define the derivative as:

$$\partial c = \frac{c_{n+1} - c_{n-1}}{2a}$$

,a being the lattice spacing,the Dirac Hamiltonian on the lattice takes the form:

$$H = H_c + H_b + H_m = i\sum(c_n^{\dagger}c_{n+1} - h.c.) - i\sum(b_n^{\dagger}b_{n+1} - h.c.) + m\sum(c_n^{\dagger}b_n + h.c.) \tag{31}$$

where we have included the mass term as well.

In arriving at (31) we have set the lattice spacing to 1/2.

The choice of a is not important in our considerations.Our results are good for any value of a. The lattice boson generators for this Hamiltonian are now written as :

$$e^c_{+l} = \sum c^\dagger_n c^\dagger_{n+l}; \;\; e^c_{-l} = -\sum c_n c_{n+l} \tag{32}$$

and

$$e^b_{+l} = \sum b^\dagger_n b^\dagger_{n+l}; \;\; e^b_{-l} = -\sum b_n b_{n+l} \tag{33}$$

The other generators h^c_i and h^b_i may be constructed in analogy with h$_i$ (defined immediately following eqn (12)). In the discussions that follow it is more convenient to consider the linear combinations

$$E^\pm_{\pm l} = \frac{1}{\sqrt{2}}(e^c_{\pm l} \pm e^b_{\pm l}) \tag{34}$$

For the case of the Dirac Hamiltonian (31) the lattice boson generators (34) are not complete, but have to include the following further ones:

$$d^1_{+l} = \sum c^\dagger_n b^\dagger_{n+l} \;\; ; d^2_{+l} = \sum b^\dagger_n c^\dagger_{n+l} \tag{35}$$

along with their conjugates , d^1_{-l} and d^2_{-l}. Once again the linear combinations

$$D^\pm_{\pm l} = \frac{1}{\sqrt{2}}(d^1_{\pm l} \pm d^2_{\pm l}) \tag{36}$$

that turn out to be of interest.

Working through the algebra we find that

$$[E^{j}_{-l}, E^{j'}_{+l'}] = 2N\delta_{ll'}\delta_{jj'} \tag{37}$$

$$[D^{j}_{-l}, D^{j'}_{+l'}] = 2N\delta_{ll'}\delta_{jj'} \tag{38}$$

$$[E^{j}_{\pm l}, D^{j'}_{\pm l'}] = 0 \tag{39}$$

where j,j' take values + and - . In the nel region we are led ,therefore , to interpret the $E^{j}_{\pm l}$ and $D^{j}_{\pm l}$ as the lattice boson generators.Note the eqn (39) follows the same approximation as the eqn(26).The operators must be normalised as in (26). The analogy follows equally well for the nfl region ,where the right-hand sides of (37) and (38) reverse the signs.Thus the superposed hole pairs created on the insulating region by the actions of E^{j}_{-l} and D^{j}_{-l} are the lattice bosons. To solve for these lattice boson modes for the Dirac Hamiltonian let us calculate the commutators of $E^{j}_{\pm l}$ and $D^{j}_{\pm l}$ with the Hamiltonian (31).These give :

$$[H_c + H_b, E^{\pm}_{l}] = 0 \tag{40}$$

$$[H_m, E^{+}_{l}] = 2mD^{+}_{l} \tag{41}$$

$$[H_m, E_l^-] = 0 \tag{42}$$

while

$$[H_c + H_b, D_l^+] = 2i[D_{l+1}^- - D_{l-1}^-] \tag{43}$$

$$[H_c + H_b, D_l^-] = 2i[D_{l+1}^+ - D_{l-1}^+] \tag{44}$$

$$[H_m, D_l^+] = 2mE_l^+ \tag{45}$$

$$[H_m, D_l^-] = 0 \tag{46}$$

The equivalent lattice boson Hamiltonian thus is:

$$H_B = 2i[\sum D_{+(l+1)}^- D_{-l}^+ + \sum D_{+(l+1)}^+ D_{-l}^-] + 2m \sum D_{+l}^+ E_{-l}^+ + h.c. \tag{47}$$

The Hamiltonian H_B is equivalent to (31) in the sense that they produce the same commutators for the lattice boson generators when we use the boson commutators (37-39).To diagonalize H_B we first carry out the transform:

$$E^{\pm}_{\pm l} = \frac{1}{\sqrt{L}} \sum E^{\pm}_{\pm}(q) e^{\mp iql} \tag{48}$$

and

$$D^{\pm}_{\pm l} = \frac{1}{\sqrt{L}} \sum D^{\pm}_{\pm}(q) e^{\mp iql} \tag{49}$$

where L is the number of independent points on the l-lattice, i.e.L=(N-1). [see the discussion below eqn. (11)]. The transformed H_B reads :

$$H_B = \sum -4sinq[D^{-}_{+}(q)D^{+}_{-}(q) + D^{+}_{+}(q)D^{-}_{-}(q)] + 2m \sum (D^{+}_{+}(q)E^{+}_{-}(q) + E^{+}_{+}(q)D^{+}_{-}(q)) \tag{50}$$

The Hamiltonian matrix in the space of $E^{\pm}_{\pm}(q)$ and $D^{\pm}_{\pm}(q)$ reads as:

$$\begin{pmatrix} 0 & a & 2m \\ a & 0 & 0 \\ 2m & 0 & 0 \end{pmatrix} \tag{51}$$

where a=-4sinq .

The above Hamiltonian matrix (51) is diagonalized to give us the eigenstates of the lattice bosons with eigenvalues 0 and $\pm\sqrt{a^2 + 2m^2}$ in the nel region.In the nfl region, because of the minus sign in(28), there is an overall minus sign for H_B of (47).Thus the eigenvalues in the nfl region just reverse their signs.

From eqn.(40) note that $E^-_{\pm l}$ are generators of symmetries of the massless Dirac Hamiltonian. Prior to the insertion of mass term, $E^{\pm}_{\pm l}$ are symmetries.The mass term keeps the $E^-_{\pm l}$ symmetries but breaks $E^+_{\pm l}$.The $E^+_{\pm l}$ mixes with $D^+_{\pm l}$.The $E^-_{\pm l}$ generate zero mass bosonic excitations that normalises the ground state. Since the $E^-_{\pm l}$ modes remain decoupled they have not explicitly appeared in our equation(47). It is important to point out that the number of independent $D^{\pm}_l$ generators differ a bit from the number of $E^{\pm}_l$.

We have shown[see below eqn.(16)] that the number of independent e_{+l} generators for the lattice of size 2N is given by N-1.For the case of the Dirac Hamiltonian of mass we have e^c_{+l} and e^b_{+l} making a total of 2(N-1) independent generators. There are , from (34) , 2(N-1) generators of type $E^{\pm}_{+l}$.

The cases of $d^{1,2}_{+l}$ are roughly the same,except that $d^{1,2}_{+0}$ are independent and non-zero.They add extra degrees. For the $D^{\pm}_{+l}$, made of the $d^{1,2}_{+l}$,eqn. (36), the $D^+_{+0} = 0$. This is because

$$D^+_{+0} = d^1_{+0} + d^2_{+0} = \sum c^\dagger_n b^\dagger_n + b^\dagger_n c^\dagger_n = 0 \tag{52}$$

from the anticommutators $\{c^\dagger_n, b^\dagger_m\} = 0$ Thus the number of D^+_l and E^+_l are identical.

The case of D^-_{+0} is as follows :

$$D^-_{+0} = d^1_{+0} - d^2_{+0} = 2\sum c^\dagger_n b^\dagger_n \tag{53}$$

This non-zero D^-_{+0} appears in (47).Thus in the hopping part of the effective boson Hamiltonian,H_B, these $D^-_{\pm 0}$ appear at the end point of the l-lattice for the D-operator.This end point is neglected in

our diagonalisation of H_B.Adding this end point ,for reasonably large values of N, does not alter our results.

5 Discussions

The fermions anticommute. This leads to fermion composites that behave as bosons on the lattice.The lattice background gives rise to the set of bosons that we call the lattice bosons. They exist near the empty lattice (nel) when the number of fermions are small.In the insulating region (nfl) these lattice bosons reappear as coherently superposed hole pairs on the insulator[9].

Since these bosons depend on the fermion algebra, they exist quite independent of the fermion-fermion interaction, or on the interactions of the fermions with the lattice background. In particular they exist even for small values of couplings.To the lowest order ,we have seen from the model of the Dirac fermions the lattice boson wavefunctions overlap,leading to the hopping over the l-lattice.This Hamiltonian is readily diagonalisable. It is to be recognised that the mass term in (50) could come from the interactions.It could arise from the fermions coupling to the Higgs boson or from parts of the QCD that lead to the dynamical mass generation and the chiral symmetry breaking [10].

To conclude ,we have shown that for fermion systems there are boson excitations that arise from the background lattice.

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